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On borderenergetic and L-borderenergetic graphs

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Abstract. A graph G of order n is said to be borderenergetic if its energy is equal to $2n-2$. In this paper, we study the borderenergetic and Laplacian borderenergetic graphs.

Keywords. energy (of graph), adjacency matrix, Laplacian matrix, signless Laplacian matrix.

1 Introduction

We first recall some definitions that will be kept throughout. Let G be a simple graph with n vertices and $m(G)$ edges, and $A(G)$ denotes its adjacency matrix. Let $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ be the Laplacian and signless Laplacian matrix of the graph G, respectively, where $D(G) = [d_{ij}]$ is the diagonal matrix whose entries are the degree of vertices, i.e., $d_{ii} = deg(v_i)$ and $d_{ij} = 0$ for $i \neq j$.

The energy of G is a graph invariant which was introduced by Ivan Gutman [6]. It is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i's are eigenvalues of G. If $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \mu_n$ and $q_1 \leq q_2 \leq \cdots \leq q_{n-1} \leq q_n$ are the Laplacian and signless Laplacian eigenvalues of G then the quantities $E_L(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m(G)}{n}|$ and $E_Q(G) = \sum_{i=1}^{n} |q_i - \frac{2m(G)}{n}|$ are called the Laplacian and signless Laplacian energy of G, respectively. Details on the properties of Laplacian and signless Laplacian energy can be found in [6, 8, 13].

The first borderenergetic graph was discovered by Hou et al. in 2001 [11], but in that time it did not attract much attention. Recently, Gong et al in [5] studied the graphs with the same

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energy as a complete graph. They put forward the concept of borderenergetic graphs.

A graph G on n vertices is said to be borderenergetic if its energy equals the energy of the complete graph K_n, i.e., if $E(G) = E(K_n) = 2(n-1)$. In [5], it was shown that there exist borderenergetic graphs on order n for each integer $n \geq 7$. The number of borderenergetic graphs were determined for $n = 7,8,9$ [5], $n = 10,11$ [14,17] and $n = 12$ [4].

In [12], a family of non-regular and non-integral borderenergetic threshold graphs was discovered. In [3], the authors obtained three asymptotically tight bounds on the number of edges of borderenergetic graphs. We refer the readers to [9,15] for more information.

An analogous concept as borderenergetic graphs, called Laplacian borderenergetic graphs was proposed in [19]. That is, a graph G of order n is Laplacian borderenergetic or L-border energetic for short, if $E_L(G) = E_L(K_n) = 2n-2$.

In [1], Deng et al. presented some asymptotically bounds on the order and size of L-borderenergetic graphs. Also, they showed that all trees, cycles, the complete bipartite graphs, and many 2-connected graphs are not L-borderenergetic. They showed in [2], a kind of threshold graphs are L-borderenergetic.

Lu et al. in [16] presented all non-complete L-borderenergetic graphs of order $4 \leq n \leq 7$ and they constructed one connected non-complete L-borderenergetic graph on n vertices for each integer $n \geq 4$, which extends the result in [20] and completely confirms the existence of non-complete L-borderenergetic graphs. Particularly, they proved that there are at least $\frac{n}{2} + 4$ non-complete L-borderenergetic graphs of order n for any even integer $n \geq 6$.

Hakimi-Nezhaad et al. in [10] generalized the concept of borderenegetic graphs for the signless Laplacian matrices of graphs. That is, a graph G of order n is signless Laplacian borderenergetic or Q-borderenergetic for short, if $E_Q(G) = E_Q(K_n) = 2n-2$. Also, they constructed sequences of Laplacian borderenergetic non-complete graphs by means of graph operations, and all the non-complete and pairwise non-isomorphic L-borderenergetic and Q-borderenergetic graphs of small order n are depicted for n with $4 \leq n \leq 9$, see Appendix.

Tao et al. in [18] considered the extremal number of edges of non-complete L-borderenergetic graph, then use a computer search to find out all the L-borderenergetic graphs on no more than 10 vertices.

Main Results

Here, we present some basic theorem used to study borderenergetic and L-borderenergetic and Q-borderenergetic graphs.

Theorem 1.1. *We have the following statements:*

1) [5]. *There are no borderenergetic graphs of order $n \leq 6$.*

2) [5]. *There exists a unique borderenergetic graph of order 7.*

3) [5]. *For any $n \geq 7$, there exist borderenergetic graphs of order n.*

4) [5]. *There are exactly 6 borderenergetic graphs of order 8.*

5) [5]. *There are exactly 17 borderenergetic graphs of order 9.*

6) [14,17]. *There are exactly 49 borderenergetic graphs of order 10.*

7) [17]. *There are exactly 158 borderenergetic graphs of order 11, of which 157 are connected.*

8) [4]. *There are exactly 572 connected borderenergetic graphs of order 12.*

8) [5]. *For each integer n $(n \geq 13)$, there exists a non-complete borderenergetic graph of order n.*

Theorem 1.2. [9]. *A borderenergetic graph of order n must possess at least $2n - 2$ edges.*

Theorem 1.3. [3]. *Let G be a k-regular integral graph of order n with t non-negative eigenvalues. If $E(G) = 2(n - t + k)$ then $E(\bar{G}) = 2(n - 1)$, where $\bar{G}$ is complement of graph G.*

Theorem 1.4. [15].

1) *There is no noncomplete borderenergetic graph with maximum degree $\triangle = 2$ or 3.*

2) *Let G be a non-complete borderenergetic graph of order n with maximum degree $\triangle = 4$. Then G must have the following properties:*

(i) $e(G) = 2n$ *or* $2n - 1$;

(ii) $|G| \leq 21$;

(iii) *G is non-bipartite;*

(iv) *the nullity, i.e., the multiplicity of eigenvalue 0, of G is 0.*

3) *Let G be a 4-regular non-complete borderenergetic graph of order n and H is a maximal bipartite subgraph of G. Then $m(G) - m(H) \geq 3$.*

Theorem 1.5. [15]. *No borderenergetic graphs have minimum degree $n - 2$. Besides, for each integer $n \geq 7$, there exists a connected noncomplete borderenergetic graph of order n with minimum degree $n - 3$ and for each even integer $n \geq 8$, there exists a noncomplete borderenergetic graph of order n with minimum degree $n - 4$.*

Theorem 1.6. *We have the following statements:*

1) [10]. *There are exactly two non-complete L-borderenergetic disconnected graphs of orders 4 and 5, respectively.*

2) [10]. *There are exactly five non-complete L-borderenergetic disconnected graphs of order 6.*

3) [10]. *There are exactly five non-complete L-borderenergetic disconnected graphs of order 7.*

4) [18]. *There are totally 18 L-borderenergetic connected graphs on less than 8 vertices.*

5) [10, 18]. *There are exactly 31 L-borderenergetic connected graphs and 27 disconnected graphs of order 8.*

6) [10, 18]. *There are exactly 16 L-borderenergetic graphs and 26 disconnected graphs of order 9.*

7) [10, 18]. *There are exactly 120 L-borderenergetic connected graphs on 10 vertices.*

Theorem 1.7. [10] *There is no Laplacian borderenergetic tree with $n \geq 3$ vertices.*

Theorem 1.8. [1]. *If G is a complete bipartite graph $K_{a,b}(1ab)$, then G is not L-borderenergetic.*

Theorem 1.9. [1]. *If G is a* 2*-connected graph with maximum degree $\triangle = 3$ and $t(G) \geq 7$ then G is not L-borderenergetic, where $t(G)$ the number of vertices of degree 3 in G.*

Theorem 1.10. [10].

1) There are no non-complete Q-borderenergetic graph of order $n \leq 5$ and 7.

2) There are exactly two non-complete Q-borderenergetic of order 6.

3) There exist exactly fourteen non-complete Q-borderenergetic graphs of order 8.

4) There exist exactly sixteen non-complete Q-borderenergetic graphs of order 9.

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Appendix

Figure 1. All Laplacian borderenergetic dis-connected graphs of order 4 and 5.

Figure 2. All Laplacian borderenergetic dis-connected graphs of order 6.

Figure 3. All Laplacian borderenergetic dis-connected graphs of order 7.

Figure 4. All Laplacian borderenergetic dis-connected graphs of order 8.

Figure 5. All Laplacian borderenergetic dis-connected graphs of order 9.